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Connection between the Green functions of the supersymmetric pair of Dirac Hamiltonians

Ekaterina Pozdeeva[1]

Department of Quantum Field Theory, Tomsk State University, 36 Lenin Avenue, Tomsk 634050, Russia

E-mail: ekatpozdeeva@mail.ru

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Abstract
The Sukumar statement about the connection between the Green functions of the supersymmetric pair of the Schrödinger Hamiltonians is generalized to the case of the supersymmetric pair of the Dirac Hamiltonians.

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1. Introduction

In recent times, a growing interest to the applications of supersymmetric quantum mechanics [1] in different fields of theoretical and mathematical physics is noticed [2–6]. Recently, a special issue of *J. Phys. A: Math. Gen.* **34** was devoted to research work in supersymmetric quantum mechanics.

It is well known that supersymmetric quantum mechanics is basically equivalent to the Darboux [7] transformation and the factorization properties of the Schrödinger equation [3, 4, 8]. The Darboux transformation method for the one-dimensional stationary Dirac equation is equivalent to the underlying quadratic supersymmetry and the factorization properties of the Dirac equation [9, 10].

Though this method is widely used for the Schrödinger equation (see, e.g. [11, 12]), its application to the Dirac equation is studied much less [13–15].

In the present paper, we generalize the Sukumar proposition [11] about the connection between the Green functions of the supersymmetric pair of the Schrödinger Hamiltonians to the case of the Dirac Hamiltonians.

In [11], Sukumar established the following integral relation:

$$\int_a^b [\tilde{G}(x, x, E) - G(x, x, E)]\, \mathrm{d}x = \frac{1}{E - E_0}, \tag{1}$$

[1] Also ICP RAS, Kosygina str. 4, 119991 Moscow, Russia.

between the Green functions $\tilde{G}, G$ corresponding to the supersymmetric pair of the Schrödinger Hamiltonians

$$H_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_0(x), \qquad LH_0 = H_1 L, \qquad H_1 = H_0 - \frac{\mathrm{d}^2 \ln f}{\mathrm{d}x^2}, \tag{2}$$

$$L = \mathrm{d}/\mathrm{d}x - f'/f, \qquad f = \psi_0(x), \qquad H_0\psi_0 = E_0\psi_0, \tag{3}$$

with the boundary conditions $\psi_n(a) = \psi_n(b) = 0$, $H_0\psi_n = E_n\psi_n$.

Above ψ_0 is the eigenfunction of initial (H_0) problem without nodes.

We would like to stress that the construction of the Darboux transformed of the Schrödinger Hamiltonian needs only one transformation function f such that $H_0 f = Ef$.

The construction of the Darboux transformation of the Dirac problem needs two spinor functions f_1, f_2 such that

$$H_0 f_1 = \lambda_1 f_1, \qquad H_0 f_2 = \lambda_2 f_2, \qquad \lambda_1 \neq \lambda_2. \tag{4}$$

Thus, four scalar functions are involved in the problem that made this problem much more complicated.

An another complication arises from the fact that the 'potential' of the Dirac problem is a matrix function. It is unclear whether it is possible to establish some relation between the Green functions of the initial and transformed Hamiltonians. But for the potential of the especial matrix structure that is described below, this is possible. Below we will consider this case.

The structure of the present paper is the following. In section 2, we give a new derivation of the Sukumar trace formula for the Schrödinger case, different from the original one. In section 3, we generalize this result for the Dirac problem with specific choice of the matrix structure of the interaction Hamiltonian and discuss the obtained results.

2. The Sukumar trace formula

In this section, we give the new derivation of the Sukumar trace formula for the Schrödinger case because the technique applied in this derivation is more easily transferred to the Dirac case. The Sukumar problem is formulated as follows.

Let us have some initial Hamiltonian

$$H_0 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_0(x), \tag{5}$$

where $V_0(x)$ is such that the set of eigenfunctions $\{X\}$ of the H_0,

$$H_0\psi_n = E_n\psi_n, \tag{6}$$

contains the subset $\{Y\} \subset \{X\}$ of this function such that $\psi_n(a) = \psi_n(b) = 0$, where $a, b (a < b)$ are some points on the real axis.

This subset of the eigenfunction $\{Y\}$ forms a complete system in the subspace $Y \subset X$ of all functions $\phi(x)$ such that $\phi(a) = \phi(b) = 0$. The last means that all $\phi(x)$ from this subspace Y can be presented in the form

$$\phi(x) = \sum_n c_n\psi_n(x). \tag{7}$$

Let us introduce the pair of the solutions of the equation

$$H_0\phi_{1,2}(x, E) = E\phi_{1,2}(x, E) \tag{8}$$

such that $\phi_1(b, E) = 0, \phi_2(a, E) = 0$. They do not belong to the subspace $Y \subset X$. Nevertheless, the following construction

$$G(x, y, E) = \frac{\phi_1(x, E)\phi_2(y, E)\Theta(x - y) + \phi_2(x, E)\phi_1(y, E)\Theta(y - x)}{W}, \tag{9}$$

$$W = \phi_1(x, E)\phi_2'(x, E) - \phi_2(x, E)\phi_1'(x, E) = \mathrm{const}(E) \tag{10}$$

obeys the inhomogeneous equation

$$(H_0 - E)G(x, y, E) = \delta(x - y) \tag{11}$$

and boundary conditions

$$G(b, y, E) = 0, \qquad G(x, a, E) = 0, \qquad x, y \in (a, b). \tag{12}$$

Thus, equation (9) represents the Green function of the Hamiltonian H_0 in the subspace Y.

Let us perform the Darboux transformation

$$\psi \to L\psi = \tilde{\psi} = \psi' - \frac{f'}{f}\psi, \qquad H_0 f = \lambda f, \tag{13}$$

$$LH_0 = H_1 L \tag{14}$$

and construct the Green function of the transformed Hamiltonian:

$$\tilde{G}(x, y, E) = \frac{\tilde{\phi}_1(x, E)\tilde{\phi}_2(y, E)\Theta(x - y) + \tilde{\phi}_2(x, E)\tilde{\phi}_1(y, E)\Theta(y - x)}{\widetilde{W}}, \tag{15}$$

$$\widetilde{W} = \tilde{\phi}_1(x, E)\tilde{\phi}_2'(x, E) - \tilde{\phi}_2(x, E)\tilde{\phi}_1'(x, E) = \widetilde{\mathrm{const}}(E). \tag{16}$$

It is simple to prove that

$$\widetilde{W} = (E - \lambda)W, \qquad \tilde{\phi}_1(E, b) = 0, \qquad \tilde{\phi}_2(E, a) = 0. \tag{17}$$

According to the Sukumar proposition

$$\int_a^b [\tilde{G}(x, x, E) - G(x, x, E)]\,\mathrm{d}x = \frac{1}{E - \lambda}. \tag{18}$$

Let us prove this statement in the manner different from the one applied in [11].

It is evident that

$$\tilde{G}(x, x, E) = \frac{\tilde{\phi}_1(x)\tilde{\phi}_2(x)}{(E - \lambda)W}. \tag{19}$$

Using definition (13), we can rewrite $\tilde{\phi}_1(x)\tilde{\phi}_2(x)$ as

$$\tilde{\phi}_1(x)\tilde{\phi}_2(x) \equiv F_1'(x) - \frac{\phi_1(x)}{f}F_2'(x), \tag{20}$$

$$\begin{aligned} F_1(x) &= \frac{\phi_1(x)}{f(x)}F_2(x), \\ F_2(x) &= f(x)\phi_2'(x) - f'(x)\phi_2(x). \end{aligned} \tag{21}$$

Taking into account that the functions f and ϕ_2 are the eigenfunctions of the initial Hamiltonian H_0 it is easy to prove that

$$-\frac{\phi_1}{f}F_2' = (E - \lambda)\phi_1\phi_2 = (E - \lambda)WG(x, x, E). \tag{22}$$

From this observation it is followed that

$$\tilde{G}(x, x, E) = G(x, x, E) + \frac{F_1'(x)}{(E - \lambda)W}. \tag{23}$$

Thus, the following expression is valid:

$$\int_a^b [\tilde{G}(x, x, E) - G(x, x, E)]\, \mathrm{d}x = \frac{1}{(E - \lambda)W}[F_1(b) - F_1(a)]. \tag{24}$$

Using definition (21), we have

$$F_1(x) = \frac{\phi_1(x)}{f(x)} F_2(x) = \phi_1(x) \left(\phi_2'(x) - \frac{f'(x)}{f(x)} \phi_2(x) \right) \tag{25}$$

$$= \phi_1(x)\tilde{\phi}_2(x) \equiv \phi_2(x)\tilde{\phi}_1(x) + W. \tag{26}$$

From this consideration and the boundary properties (17) of the $\tilde{\phi}_{1,2}(x)$, it is followed that

$$F_1(b) - F(a) = W \tag{27}$$

and

$$\int_a^b [\tilde{G}(x, x, E) - G(x, x, E)]\, \mathrm{d}x = \frac{1}{E - \lambda}, \tag{28}$$

what is just the Sukumar trace formula.

This relation implies that the difference of the initial Green function and Darboux-transformed one is finite even if the trace of one of these functions is infinite what is the case, for instance, for oscillator Hamiltonian (pure discrete spectrum) or any Hamiltonian with continuous spectrum [12]. In the first case (the oscillator Hamiltonian), the trace of the difference has the simple form (pole term). The case of continuous spectrum is much more complicated and has been discussed in detail in [12].

3. The Dirac problem

In some cases, the one-dimensional four-component Dirac equation admits the two-component representation:

$$H_0\psi(x) = E\psi(x), \qquad \psi^T(x) = (\psi_1(x), \psi_2(x)), \tag{29}$$

$$H_0 = \mathrm{i}\sigma_2\partial_x + \sigma_3(m + S(x)) + \sigma_1 U(x) + V(x), \tag{30}$$

where $\sigma_{1,2,3}$ are the Pauli matrices and $S(x)$, $U(x)$ and $V(x)$ are the real functions of x.

Such a Hamiltonian can, for example, describe the interaction with the external electrostatic field $\Phi(x)$ of the spin one-half particle with the charge e, the anomalous atom magnetic moment λ and the position-dependent mass $m(x) = m + S(x)$.

In this case

$$V(x) = e\Phi(x), \tag{31}$$

$$U(x) = \lambda\Phi'(x). \tag{32}$$

The interaction with the external electrostatic field $\Phi(x)$ of neutral ($e = 0$) particle with a fixed mass $m(x) = \mathrm{const} = m$ is described by Hamiltonian (30) with $S(x) = V(x) = 0$.

In this paper, we will deal only with this case because of its comparable simplicity. The general case $S(x) \neq 0$, $V(x) \neq 0$ is much more complicated for the consideration and we plan to discuss it in a separate paper.

So, we look for the solution of the Dirac equation

$$H_0\psi = E\psi, \qquad \psi = (\psi_1, \psi_2)^T, \tag{33}$$

$$H_0 = \mathrm{i}\sigma_2\partial_x + m\sigma_3 + U\sigma_1 \tag{34}$$

with the boundary conditions. These conditions can be of two types:

$$(i) \quad \psi_1(a, E) = \psi_1(b, E) = 0, \tag{35}$$

$$(ii) \quad \psi_2(a, E) = \psi_2(b, E) = 0. \tag{36}$$

Let us rewrite the Dirac equation in the component form:

$$\psi_2' + U\psi_2 = (E - m)\psi_1, \tag{37}$$

$$-\psi_1' + U\psi_1 = (E + m)\psi_2. \tag{38}$$

In what follows, it is essential that in the case (35) among other solutions of the Dirac equation exist the especial solution of the form

$$\psi_1(x) \equiv 0, \qquad \psi_2(x) = \exp\left(-\int^x U(x')\,\mathrm{d}x'\right), \qquad E = -m, \tag{39}$$

and in the case (36)

$$\psi_2(x) \equiv 0, \qquad \psi_1(x) = \exp\left(\int^x U(x')\,\mathrm{d}x'\right), \qquad E = m. \tag{40}$$

First of all, consider the Dirac equation with the boundary conditions (35). This condition chooses among all solutions of the eigenvalue problem (33) the subset of solutions with discrete spectrum

$$H_0\psi^{(n)} = E_n\psi^{(n)}, \qquad \psi_1^{(n)}(a) = \psi_1^{(n)}(b) = 0. \tag{41}$$

The solution

$$H_0 v = -mv \tag{42}$$

belongs to this subset.

Let us denote by u the solution of the system of equations (37) and (38) with the eigenvalue E that is one of the neighboring on $E = -m$. Thus, we have two possibilities: $E > -m$ and $E < -m$. Denote the eigenvalue of the chosen solution by λ: $h_0 u = \lambda u$.

Consider the transformation matrix

$$\hat{u} = \begin{pmatrix} v_1 & u_1 \\ v_2 & u_2 \end{pmatrix}, \qquad v_1 \equiv 0, \qquad v_2 = \exp\left(-\int^x U(x')\,\mathrm{d}x'\right) \tag{43}$$

and construct the Darboux transformation operator of the form

$$L = \partial_x - \hat{u}_x u^{-1}, \qquad \hat{u}_x = \partial_x\hat{u}. \tag{44}$$

Define the transformed Hamiltonian H_1 with the help of intertwining relation

$$LH_0 = H_1 L. \tag{45}$$

It can be proved [9] that

$$H_1 = H_0 + [\mathrm{i}\sigma_2, \hat{u}_x\hat{u}^{-1}] = \mathrm{i}\sigma_2\partial_x + \lambda\sigma_3 - (\ln f)'\sigma_1, \tag{46}$$

where $f = u_1(x)$.

If $\psi(x)$ is the solution of (33), then $\tilde{\psi} = L\psi$ is the solution of the equation

$$H_1\tilde{\psi} = E\tilde{\psi} \tag{47}$$

with the same value of energy E.

With the help of a simple calculation, it is easy to obtain

$$\tilde{\psi}_1 = \psi_1' - \frac{f'}{f}\psi_1, \qquad f = u_1, \tag{48}$$

$$\tilde{\psi}_2 = (E-\lambda)\psi_1. \tag{49}$$

Let us denote by $\psi^{(L)}$, $\psi^{(R)}$ the solution of the initial equation (33) that obeys the following boundary conditions:

$$\psi_1^{(L)}(a) = 0, \qquad \psi_1^{(R)}(b) = 0. \tag{50}$$

Then from equations (48) and (49) it is easily obtained that

$$\tilde{\psi}_{1,2}^{(L)}(a) = 0, \qquad \tilde{\psi}_{1,2}^{(R)}(b) = 0. \tag{51}$$

Let us construct the Green functions of the initial (H_0) and the transformed (H_1) Hamiltonians. These functions can be expressed in terms of the $\psi^{(L)}$, $\psi^{(R)}$ (for h_0) and $\tilde{\psi}^{(L)}$, $\tilde{\psi}^{(R)}$ (for H_1), respectively,

$$\begin{gathered} G_0(x,y,E) = \frac{\psi^{(R)}(x)\psi^{(L)T}(y)\Theta(x-y) + \psi^{(L)}(x)\psi^{(R)T}(y)\Theta(y-x)}{W}, \\ W = \psi_1^{(R)}(x)\psi_2^{(L)}(x) - \psi_1^{(L)}(x)\psi_2^{(R)}(x) = \mathrm{const}(E) \end{gathered} \tag{52}$$

and similarly,

$$\begin{gathered} G_1(x,y,E) = \frac{\tilde{\psi}^{(R)}(x)\tilde{\psi}^{(L)T}(y)\Theta(x-y) + \tilde{\psi}^{(L)}(x)\tilde{\psi}^{(R)T}(y)\Theta(y-x)}{\widetilde{W}}, \\ \widetilde{W} = \tilde{\psi}_1^{(R)}(x)\tilde{\psi}_2^{(L)}(x) - \tilde{\psi}_1^{(L)}(x)\tilde{\psi}_2^{(R)}(y) = (E-\lambda)(E+m)W. \end{gathered} \tag{53}$$

Let us introduce the quantities

$$I_0(x,E) = \mathrm{tr}\, G_0(x,x,E) = \frac{\psi_1^{(L)}(x)\psi_1^{(R)}(x) + \psi_2^{(L)}(x)\psi_2^{(R)}(x)}{W}, \tag{54}$$

$$I_1(x,E) = \mathrm{tr}\, G_1(x,x,E) = \frac{\tilde{\psi}_1^{(L)}(x)\tilde{\psi}_1^{(R)}(x) + \tilde{\psi}_2^{(L)}(x)\tilde{\psi}_2^{(R)}(x)}{\widetilde{W}}. \tag{55}$$

Now let us express the numerator of the $I_1(x,E)$ in terms of the functions $\psi^{(L)}$, $\psi^{(R)}$:

$$\tilde{\psi}_1^{(L)}(x)\tilde{\psi}_1^{(R)}(x) + \tilde{\psi}_2^{(L)}(x)\tilde{\psi}_2^{(R)}(x) = A + B, \tag{56}$$

$$A = \left(\psi_1^{'(L)} - \frac{f'}{f}\psi_1^{(L)}\right)\left(\psi_1^{'(R)} - \frac{f'}{f}\psi_1^{(R)}\right), \tag{57}$$

$$B = (E-\lambda)^2\psi_1^{(L)}\psi_1^{(R)}. \tag{58}$$

Using the similar trick as above (see section 2), one can obtain

$$A = F_1' - \frac{\psi_1^{(L)}}{f}F_2', \tag{59}$$

$$F_1 = \psi_1^{(L)} \left(\psi_1^{'(R)} - \frac{f'}{f} \psi_1^{(R)} \right), \tag{60}$$

$$F_2 = f \psi_1^{'(R)} - f' \psi_1^{(R)}. \tag{61}$$

Using the Dirac equations (37) and (38) in order to exclude the derivatives $\psi^{('R)}$ and f' from the expression for F_2 and once more for the excluding derivatives from the expression for F_2' after simple but rather cumbersome calculations, one can obtain

$$-\frac{\psi_1^{(L)}}{f} F_2' = (E^2 - \lambda^2) \psi_1^{(L)} \psi_1^{(R)}. \tag{62}$$

As a result, we get

$$A + B = F_1' + 2E(E - \lambda) \psi_1^{(L)} \psi_1^{(R)}. \tag{63}$$

Let us now consider the difference

$$\Delta I = I_1 - I_0 = \operatorname{tr} G_1(x, x, E) - \operatorname{tr} G_0(x, x, E). \tag{64}$$

With the use of (54) and (55), we have

$$\Delta I = \frac{F_1'}{(E - \lambda)(E + m)W} + \frac{(E - m)\psi_1^L \psi_1^{(R)} - (E + m)\psi_2^{(L)} \psi_2^{(R)}}{(E + m)W}. \tag{65}$$

Again by the use of the Dirac equation for $\psi^{(L,R)}$, it is easy to obtain

$$(E - m)\psi_1^L \psi_1^R - (E + m)\psi_2^{(L)} \psi_2^{(R)} = \left(\psi_1^{(L)} \psi_1^{(R)}\right)' = F_3'. \tag{66}$$

Taking into account the identity

$$\psi_1^{(L)} \psi_2^{(R)} = \psi_1^{(R)} \psi_2^{(L)} + W \tag{67}$$

and the boundary conditions

$$\psi_1^{(L)}(a) = \psi_1^{(R)}(b) = 0, \tag{68}$$

it is easy to see that

$$\int_a^b F_3' \, \mathrm{d}x = W. \tag{69}$$

Let us briefly discuss the quantity

$$F_1(x) = \psi_1^{(L)} \left(\psi_1^{'(R)} - \frac{f'}{f} \psi_1^{(R)} \right) \equiv \psi_1^{(L)} \tilde{\psi}_1^{(R)}. \tag{70}$$

It can be rewritten as follows:

$$F_1(x) = \psi_1^{(L)} \psi_1^{'(R)} - \psi_1^{'(L)} \psi_1^{(R)} + \psi_1^{(R)} \tilde{\psi}_1^{(L)}, \tag{71}$$

that after excluding the derivatives from the last expression with the help of the Dirac equation can be presented in the form

$$F_1(x) = (E + m)W + \psi_1^{(R)} \tilde{\psi}_1^{(L)}. \tag{72}$$

Taking into account this observation, it is easy to obtain

$$\int_a^b F_1' = (E + m)W. \tag{73}$$

Combining (73) with (69), we obtain the final result:

$$\int_a^b [\operatorname{tr} G_1(x, x, E) - \operatorname{tr} G_0(x, x, E)] \, \mathrm{d}x = \frac{1}{E - \lambda} + \frac{1}{E + m}. \tag{74}$$

In the case (36), the similar calculations lead to the relation:

$$\int_a^b [\operatorname{tr} G_1(x,x,E) - \operatorname{tr} G_0(x,x,E)]\, \mathrm{d}x = \frac{1}{E-\lambda} + \frac{1}{E-m}. \tag{75}$$

This is the generalization of the Sukumar trace formula for the Dirac case, which takes into account the above restriction $V(x) = S(x) = 0$.

It can be supposed that in the case $V(x) \neq 0$, $S(x) \neq 0$ the trace formula reads

$$\int_a^b [\operatorname{tr} G_1(x,x,E) - \operatorname{tr} G_0(x,x,E)]\, \mathrm{d}x = \frac{1}{E-\lambda_1} + \frac{1}{E-\lambda_2}, \tag{76}$$

where $\lambda_{1,2}$ are defined by (4), i.e. for enough general case $\lambda_1 \neq \pm m$.

The all said about the implication of the Sukumar result (1) (see the end of section 1) is also valid in the case under consideration.

It is well known that the Green function of any Hamiltonian can be represented in the form

$$G(x,y,E) = \Sigma_n \frac{\psi_n(x)\psi_n^T(y)}{E_n - E}, \tag{77}$$

$$\int \operatorname{tr}(\psi_n(x)\psi_n^T(x))\, \mathrm{d}x = 1 \tag{78}$$

(the so-called *spectral representation*).

If the integral

$$I = \int \operatorname{tr} G(x,x,E)\, \mathrm{d}x \neq \pm\infty, \tag{79}$$

then

$$I = \sum_n \frac{1}{E_n - E} \tag{80}$$

and the Sukumar and our results are obvious, since they reflect the well-known observation that the Darboux transformation delete, from the spectrum of the initial Hamiltonian, the terms corresponding to the states that are used for the construction of the transformation matrix u.

In the case when $|I| = \infty$, this result is not trivial as the difference of two infinities is not well define. Nevertheless, the use of alternative representations (9), (15), (52) and (53) for the Green functions allows us to resolve this problem.

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